MA 234, Spring 2016

Stuff to know for the Final.

Exam will be Closed Book (no books, no notes, no collaboration, no electronic communication during exam). Calculators will be **not** allowed.

Note that the list below is for convenience purposes. Actual class/homework material[1] takes precedence in case of differences.

Emphasis will be on the material of parts (13)–(14) on this list.

(1) *Basic things. Sections 1.2–1.5.* Notion of complex numbers. Arithmetic operations. Complex conjugation. Complex plane. Polar (trigonometric, exponential) representation of a complex number. Absolute value of a complex number, its behavior under arithmetic operations. Argument $\arg z$ of a complex number. Principal argument $\mathrm{Arg}\, z$ of a complex number. Converting between cartesian and polar form. Multiplication of complex numbers in polar coordinates, geometric interpretation of multiplication. DeMoivre's formula. Roots on degree n of a complex number.

(2) *Sets on a complex plane. Section 1.6.* Curves, parameterizations, smooth curves. Parameterization of a straight line, of a circle. Open disk as an ε-neighborhood, closed disk, punctured disk on a complex plane. Interior, exterior, boundary points of a set. Open sets. Closed sets. Connected sets. Domains and regions. Bounded and unbounded sets.

(3) *Complex functions: basics. Section 2.1.* Notion of complex function, synonymic terminology. Domain, range. Cartesian and polar forms of a complex function. Image of a set, inverse image. One-to-one functions. Functions "onto" a set. Linear function, geometric interpretation. Basic techniques for finding image of a set.

(4) *Complex functions: some specific mappings. Sections 2.2, 2.5.* Mapping z^2: basic properties, image of vertical and horizontal straight lines and regions bounded by those. Principal square root function $z^{1/2}$: basic properties, image of vertical and horizontal straight lines and regions bounded by those. Basic properties of z^n and the principal nth root function $z^{1/n}$. Reciprocal function $\frac{1}{z}$: basic properties, image of straight lines, circles (passing vs not passing through 0), and regions bounded by those.

(5) *Limits of functions and continuity. Sections 2.3, 2.4.* Limit of a function, relation to real-value limits. Arithmetic properties of limit. Basic techniques for finding limits. Directional limit, relation to limit of a complex function. Continuous functions, relation to continuity of Re and Im. Extended complex plane and stereographic projection. $\frac{1}{z}$ as a mapping of the extended complex plane. Notion of a branch of a multi-valued function. Branches of $z^{1/2}$. Basic idea of a Riemann surface. Riemann surface for $z^{1/2}$.

(6) *Differentiable and analytic functions. Sections 3.1-3.3.* Definition of the derivative of a complex function. Finding derivative by a direct computation. Rules of differentiation. Cauchy–Riemann equations, Cauchy–Riemann conditions. Analytic functions (at a point, on a domain). Notion of a harmonic function, relation of complex analytic functions to harmonic functions. Harmonic conjugate, finding a harmonic conjugate.

[1] Read this as: *instructor's recollection* of actual class/homework material.

(7) *Sequences and series. Sections 4.1, 4.3–4.4.* Sequences of complex numbers. Limit of a sequence, basic properties of limits of sequences. Complex series, sum of series. Geometric series. Ratio test, root test. Notion of limit superior. Power series, disc and radius of convergence of a power series. Finding radius of convergence by Cauchy–Hadamard root formula, by d'Alembert's ratio formula. Infinite differentiability of the sum of a power series.

(8) *Elementary functions: exp, log, power function. Sections 5.1–5.3.* Definition of complex exponential function e^z. Main properties of e^z. e^z as a mapping. Expression of e^z as $u + iv$, polar expression of e^z. Complex logarithm (multivalued function $\log z$ and principal value $\operatorname{Log} z$). Expression of $\operatorname{Log} z$ as $u + iv$. Main properties of complex logarithm. Complex power function z^c: definition, main properties, principal value. Differentiability and derivatives of $e^z, \operatorname{Log} z, z^c$.

(9) *Elementary functions: trig and hyperbolic functions. Sections 5.4–5.5.* Definitions of complex $\cos z$ and $\sin z$: through power series and through the exponential. Main properties and formulas for $\cos, \sin$. Properties that carry from real line; properties that do not carry from real line. Expression of $\sin, \cos$ as $u + iv$. Same treatment of hyperbolic functions $\cosh z$, $\sinh z$. Definition and basic properties of $\tan z$, $\cot z$, $\tanh z$, $\coth z$. Inverse trig and hyperbolic functions, their expression through log and $\sqrt{\ }$. Differentiability and derivatives of trig, hyperbolic, and their inverse functions.

(10) *Complex integrals: basics. Sections 6.1–6.2.* Integral of a complex function along a real line segement, two ways of computing. Basic properties. Notion of a contour. Definition of a contour integral. Expression of integral through parameterization of contour. Basic properties of complex contour integrals. Integral $\int_C dz/(z-a)$ along a circle C centered at a. ML-inequality.

(11) *Complex integrals: Cauchy–Goursat theorem. Section 6.3–6.4.* Cauchy–Goursat theorem, its other versions: Deformation of contours theorem, Extended Cauchy–Goursat theorem. Computing integrals using Cauchy–Goursat theorem. Path independence and indefinite integrals (antiderivatives). Fundamental theorem of calculus (definite integrals). Using Fundamental theorem of calculus to compute integrals.

(12) *Complex integrals: Cauchy integral formula. Section 6.5–6.6.* Cauchy integral formula. Cauchy integral formula for derivatives. Using Cauchy integral formula (and that for derivatives) to compute integrals. Consequences of Cauchy integral formula: Morera's theorem, Liouville's theorem, Gauss's mean value theorem, Maximum modulus principle, Cauchy inequalities, fundamental theorem of algebra.

(13) *Taylor, Laurent series, singular points. Sections 7.2–7.5.* Taylor and Maclaurin series of an analytic function. Taylor's theorem. Different methods for finding Taylor series or its first few terms. Taylor series of notable functions (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, $1/(1+z)$, $\operatorname{Log} z$). Laurent series. Laurent series, annulus of convergence. Laurent's theorem. Different methods for finding Laurent's series of a function. Isolated singular points and their classification (removable, pole of order N, essintial singularities). Behavior of poles and zeros under multiplication of functions. Limit of a function at singular points.

(14) *Residues, Residue theorem, and their applications. Sections 8.1–8.4, 8.7.* Notion of residue of a function at a point. Cauchy's residue theorem. Using Cauchy theorem to compute complex integrals. Application of residues to real variable integrals: trigonometric integrals, integrals of rational functions $P(x)/Q(x)$ over real line, integrals of $P(x)\cos x/Q(x)$, $P(x)\sin x/Q(x)$ over real line. Argument principle and Rouché theorem. Using Rouché theorem to find number of zeros of a function in a given region.